MANIFOLDS NEAR THE BOUNDARY
OF EXISTENCE

KARSTEN GROVE & PETER PETERSEN V

Our primary purpose is to study relationships between bounds on sectional curvature K_M, diameter d_M, and volume V_M, together with their effects on the topology of closed Riemannian n-manifolds M. For purposes of illustration, we normalize for the moment all manifolds so that $d_M = \pi$. Consequently, $\min K_M \leq 1$ by the Bonnet-Meyers theorem, and from the Rauch comparison theorem, $V_M \leq V(n, \min K_M, \pi)$, where $V(n, k, D)$ denotes the volume of a D-ball, $B_k^n(\bar{p}, D)$ in the n-dimensional simply connected manifold of constant curvature k. We may thus represent any closed Riemannian n-manifold as a point in the $(\min K_M, V_M)$-plane (Figure 1, next page).

Any manifold to the right of a vertical line has *a priori* bounded Betti-numbers as Gromov proved in [6]. Moreover, in regions bounded to the left by a vertical line and below by a horizontal line above the axis, only finitely many homotopy types occur [10]. In fact, at least when $n \neq 3, 4$, such regions contain at most finitely many diffeomorphism types [15]. Understanding convergence and limit spaces with respect to the Gromov-Hausdorff distance is an essential tool in the proof of the latter result.

On the basis of these results, it is natural to examine the topological properties of manifolds close to the boundary of existence. As already explained, the existence region is bounded above by the curve $V(n, \cdot, \pi)$, to the right by the line $\min K_M = 1$, and below by the $\min K_M$-axis. Only two points, $(1, V(n, 1, \pi))$ and $(1/4, V(n, 1/4, \pi))$, on this boundary are actually represented by manifolds, namely the sphere and the real projective space of constant curvature. Moreover, any manifold located strictly between the vertical lines through these two extremal points is homeomorphic to S^n [14].
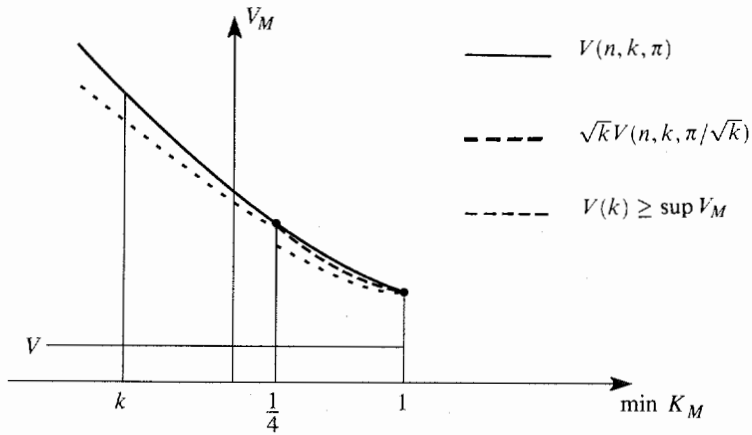
FIGURE 1

One of our main results shows, in particular, that only uniform collapse can occur near the $\min K_M$-axis. In fact, we have

Theorem A. *Let X be an accumulation point of a sequence $\{M_i\}$ of closed, connected n-manifolds with $K_{M_i} \geq k$ and $d_{M_i} \leq D$. Then*

(i) *X is a length space with Toponogov curvature $K_X \geq k$ and diameter $d_X \leq D$,*

(ii) *$\dim X = m \leq n$ and X has dimension m at every point,*

(iii) *$\dim \partial O \geq m - 1$ for any open nondense subset $O \neq \varnothing$ in X,*

(iv) *X is a Cantor manifold.*

Note that, in general, any polyhedral compact subset of a Euclidean space is the Gromov-Hausdorff limit of a sequence of closed Riemannian manifolds.

At the other extreme, we prove that there is an a priori upper bound for the volume better than the one given by the Rauch comparison theorem (see Figure 1).

Theorem B. *Let M be a closed, connected n-manifold with $K_M \geq k$ and $d_M \leq D$. Then*

$$V_M \leq V(n, k, D) - \varepsilon(n, k, D),$$

where $\varepsilon > 0$ except when $k > 0$ and $D \in \{\frac{1}{2}\pi/\sqrt{k}, \pi\sqrt{k}\}$. When $k > 0$ and $D \in (\frac{1}{2}\pi/\sqrt{k}, \pi/\sqrt{k})$, $V(n, k, D)$ can be replaced by $(D\sqrt{k}/\pi)V(n, k, \pi/\sqrt{k})$, $\varepsilon(n, k, D) > 0$, and $\limsup \varepsilon > 0$ when $D \to \frac{1}{2}\pi/\sqrt{k}$.

This generalizes and extends a result obtained independently in [21] (cf. Theorem 1.4). It is also related to a generalization of a conjecture due to Aleksandrov [1].

In left-sided neighborhoods of the extremal points S^n (respectively $\mathbf{R}P^n$) only manifolds diffeomorphic to S^n (respectively $\mathbf{R}P^n$) occur [17] (cf. also Remark 3.3). With straightforward modifications of [11] one proves that any manifold M^n with $K_M \geq k$, $d_M \leq D$, and V_M sufficiently close to $V(n, k, D)$ has the homotopy type of $\mathbf{R}P^n$ unless $k > 0$ and $d > \frac{1}{2}\pi/\sqrt{k}$. Thus, it is natural to conjecture that manifolds to the left of the extremal $\mathbf{R}P^n$ and with near maximal volume look topologically like $\mathbf{R}P^n$.

We point out that the conclusion in Theorem B most likely is valid with $K_M \geq k$ replaced by $\text{Ric}_M \geq (n - 1)k$ and $K_M \geq k'$ for any $k' \leq k$. This is exemplified by Yamaguchi's generalization of [17] in [22]. Here and in other problems concerning manifolds M with Ricci curvature bounded from below, an additional assumption on $\min K_M$ may seem merely technical. Keeping [10], [15], and our present paper in mind, we feel, however, that any $\min K_M$ bound on a class of manifolds $\mathcal{M}$ restrict the geometry and topology of elements in the Gromov-Hausdorff closure $\overline{\mathcal{M}}$ so much that an additional strong geometric assumption, say of Ric_M, frequently will yield the desired rigidity or structure properties (cf. [12]).

The paper is organized in three sections and one appendix. In §1 we present a general volume comparison theorem which is used in the proof of Theorem B as well as to simplify the proof of Main Lemma 1.3 in [10] (cf. 1.5). §2 is devoted to the study of limit spaces X of manifolds with a lower curvature bound k, and an upper diameter bound D. In particular, it is shown that the Toponogov triangle comparison theorem is valid in X, and for any $p \in X$ we exhibit exponential maps from star-shaped subsets in $\overline{B}_k^n(\overline{p}, D)$ onto X. This is used in the proofs of Theorems A and B. The latter is given in §3. The appendix is concerned with a generalization of the Arzela-Ascoli theorem to families of maps between different spaces. This is used in our construction of exponential maps in §2.

For basic results and notion from Riemannian geometry and dimension theory which will be used freely, we refer to [2], [9], and [16].

1. Volume comparison

In this section we will extend the volume comparison theorems in [11] and [3] for non-star-shaped sets. Among immediate applications we give

an estimate for the maximal volume of manifolds M with $K_M \geq 1$ and $d_M \geq \frac{\pi}{2}$.

Let M be a compact, connected n-dimensional Riemannian manifold with $K_M \geq k$ and $d_M \leq D$ (if $k > 0$ we assume $D < \pi/K_M$), and fix a point $p \in M$. Similarly, let $\overline{M}$ be the simply connected n-dimensional space form of curvature k, and fix $\overline{p} \in \overline{M}$. For any subset $Q \subset M$ denote by $\Gamma_{pQ} = \bigcup \Gamma_{pq}$, $q \in Q$ the set of all minimal geodesics from p to all points $q \in Q$ parameterized on $[0,1]$. The corresponding set of initial velocity vectors will be denoted by $\dot{\Gamma}_{pQ} \subset T_p M$. Identifying $T_p M$ with $T_{\overline{p}} \overline{M}$, we let $\overline{Q} = \exp_{\overline{p}}(\dot{\Gamma}_{pQ}) \subset \overline{M}$. Then by construction, $\dot{\Gamma}_{\overline{p}\overline{Q}} = \dot{\Gamma}_{pQ} \subset T_{\overline{p}} \overline{M}$.

Lemma 1.1. *With the notation above, consider functions F, $G: Q \times [0, \infty) \to [0, \infty)$ and via $\exp_p \circ \exp_{\overline{p}}^{-1}$ corresponding functions $\overline{F}$, $\overline{G}: \overline{Q} \times [0, \infty) \to [0, \infty)$, where $G(q, \cdot)$ is nondecreasing for all $q \in Q$. Then for any $R \leq D$, the sets*

$$H = \{x \in \overline{B}(p, R) | F(q, d(x, p)) \leq G(q, d(x, q)), q \in Q\},$$
$$\overline{H} = \{\overline{x} \in \overline{B}_k^n(\overline{p}, R) | \overline{F}(\overline{q}, d(\overline{x}, \overline{p})) \leq \overline{G}(\overline{q}, d(\overline{x}, \overline{q})), \overline{q} \in \overline{Q}\}$$

are related by $\operatorname{vol} H \leq \operatorname{vol} \overline{H}$.

Proof. Let $v \in \dot{\Gamma}_{pH}$ i.e., $c_v(t) = \exp_p(tv)$, $t \in [0, 1]$ is a minimal geodesic from p to $x = \exp_p(v) \in H$. Using our identification $T_p M = T_{\overline{p}} \overline{M}$, and let $\overline{x} = \exp_{\overline{p}}(v)$. Then for $\overline{q} = \exp_{\overline{p}}(u)$ and $q = \exp_p(u)$, $u \in \dot{\Gamma}_{pQ} = \dot{\Gamma}_{\overline{p}\overline{Q}}$, we have

$$\overline{F}(\overline{q}, d(\overline{x}, \overline{p})) = F(q, d(\overline{x}, \overline{p})) = F(q, d(x, p))$$
$$\leq G(q, d(x, q)) \leq G(q, d(\overline{x}, \overline{q})) = \overline{G}(\overline{q}, d(\overline{x}, \overline{q})).$$

The second inequality follows from the Toponogov distance comparison theorem and the assumption on $G(q, \cdot)$. This proves that $\dot{\Gamma}_{pH} \subset \dot{\Gamma}_{\overline{p}\overline{H}}$ and thus $H \subset \exp_p \circ \exp_{\overline{p}}^{-1}(\overline{H})$. In particular, $\operatorname{vol}(H) \leq \operatorname{vol}(\exp_p \circ \exp_{\overline{p}}^{-1}(\overline{H})) \leq \operatorname{vol} \overline{H}$, where the last inequality follows from the Rauch comparison theorem.

Examples 1.2. (a) $H = B(p, R)$, the R-ball centered at p, when $F(q, t) = t$, and $G(q, t) = R$ for all $(q, t) \in Q \times [0, \infty)$.

(b) $H = M - B(Q, R)$, the complement of the R-tube around Q, when $F(q, t) = R$ and $G(q, t) = t$ for all $(q, t) \in Q \times [0, \infty)$ (cf. [11]).

(c) $H = \{x \in M | d(x, p) \leq d(x, Q)\}$, the half-space containing p determined by $\{p\}$ and Q, when $F(q, t) = G(q, t) = t$ for all $(q, t) \in Q \times [0, \infty]$.

(d) With F, G as in (c) we get in particular using 1.1,

$$\mathrm{vol}(B(p_1, R) \cup B(p_2, R)) \le \mathrm{vol}(B_k^n(\overline{p}_1, R) \cup B_k^n(\overline{p}_2, R)),$$

where $d(\overline{p}_1, \overline{p}_2) = d(p_1, p_2)$, $\overline{p}_1$, $\overline{p}_2 \in \overline{M}$.

Remark 1.3. If we weaken the curvature assumption $K_M \ge k$ in 1.1 to $\mathrm{Ric}_M \ge (n-1)k$ and $K_M \ge k'$ for some $k' < k$, we obtain a similar volume comparison. The only difference is that one replaces $\overline{G}$ by $\overline{\overline{G}}(q, t) = \overline{G}(\overline{q}, c \cdot t)$ for a suitable constant $c > 0$ depending only on D, k, and k'. Observe that this change does not affect $\overline{H}$ in 1.2(a), which is then exactly Bishop's volume comparison theorem.

We will use 1.1 in full generality to prove Theorem B of the introduction (cf. §3). Here we give two applications of the half-space version 1.2(c).

Theorem 1.4. *Suppose M satisfies $K_M \ge k > 0$ and $d_M > \frac{1}{2}\pi/\sqrt{k}$. Then $V_M \le (d_M \cdot \sqrt{k}/\pi)\,\mathrm{vol}\,S^n(1/\sqrt{k}) < V(n, k, d_M)$.*

Proof. For convenience, we let $k = 1$. For any pair of points p, $q \in M$, $V_M = \mathrm{vol}\,H_1 + \mathrm{vol}\,H_2$, where $H_1 = \{x \in M | d(x, p) \le d(x, q)\}$ and $H_2 = \{x \in M | d(x, p) \ge d(x, q)\}$. Assume p is a critical point for q, i.e., the unit vectors in the directions $\dot{\Gamma}_{pq}$ determine a weak $\frac{\pi}{2}$-net in $S_p(1) \subset T_pM$ (cf. [6], [14]). From the appendix of [10] we conclude $\mathrm{vol}\,\overline{H}_1 \le \mathrm{vol}\{\overline{x} \in S^n(1) | d(\overline{x}, \overline{p}) \le d(\overline{x}, \{\overline{q}_1, \overline{q}_2\})\} = ((d(p, q)/(2\pi))\,\mathrm{vol}\,S^n(1)$, where $d(\overline{p}, \overline{q}_1) = d(\overline{p}, \overline{q}_2) = d(p, q)$ and $\overline{p}$, $\overline{q}_1$, $\overline{q}_2$ lie on the same great circle. In case $d(p, q) = d_M$, p and q are critical points for each other, whence $\mathrm{vol}\,H_i \le ((d_M)/(2\pi))\,\mathrm{vol}\,S^n(1)$, $i = 1, 2$.   q.e.d.

A different proof of this theorem has been given independently in [21].

The idea of estimating the volume of half spaces also simplifies the proof of Main Lemma 1.3 in [10], and yields more explicit and better *a priori* constants. We will use the notation from [10] without further comments.

Lemma 1.5. *Let M be a closed connected Riemannian n-manifold with $k_M \ge k$, $d_M \le D$, and $V_M \ge v > 0$. For any $\alpha > 0$ with $\alpha < v \cdot \frac{\pi}{2}V(n, k, D)^{-1}$, there is an $r > 0$ such that $d(p, q) \ge r$ whenever p, $q \in M$ are points where the directions of the set of minimal geodesics between p and q form $(\frac{\pi}{2} + \alpha)$-nets in $S_p(1)$ and $S_q(1)$.*

Proof. Fix $\alpha > 0$ so that $\frac{\alpha}{2}V(n, k, D) < \frac{1}{2}v$. Assume without loss of generality that the directions of a finite set of minimal geodesics between p and q form $(\frac{\pi}{2} + \alpha)$-nets in $S_p(1) \subset T_pM$ and $S_q(1) \subset T_qM$. Again $V_M = \mathrm{vol}\,H_1 + \mathrm{vol}\,H_2$, where H_i, $i = 1, 2$, are the half spaces determined by p and q. Thus $V_M \le \mathrm{vol}\,\overline{H}_1 + \mathrm{vol}\,\overline{H}_2$, where $\overline{H}_1$ are corresponding half spaces in a D-ball $\overline{B}_k^n(\overline{p}, D)$ in constant curvature. As in the proof

of 1.4 we use [10, Appendix] to get the estimate

$$\operatorname{vol}\overline{H}_i \le \operatorname{vol}\{\overline{x} \in \overline{B}_k^n(\overline{p}, D) | d(\overline{x}, \overline{p}) \le d(\overline{x}, \{\overline{q}_1, \overline{q}_2\})\},$$

where $d(\overline{p}, \overline{q}_1) = d(\overline{p}, \overline{q}_2) = d(p, q)$ and now the triangle $(\overline{q}_1, \overline{p}, \overline{q}_2)$ has angle $(\pi - 2\alpha)$ at $\overline{p}$. Therefore, $\operatorname{vol}\overline{H}_i < \frac{1}{2}v$, $i = 1, 2$, for $d(p, q)$ sufficiently small.

2. Limits of manifolds curved from below

The primary objective of this section is to begin an investigation of spaces X in the Gromov-Hausdorff closure of the class $\mathscr{M}(n, k, D)$ consisting of closed connected Riemannian n-manifolds M, with $K_M \ge k$, $d_M \le D$. Such X are in particular length spaces (cf. [8]).

Throughout this and the next section, when $X = \lim M_i$, $M_i \in \mathscr{M}(n, k, D)$, we fix a metrix on $X \coprod_i M_i$ so that M_i converges to X inside this space in the classical Hausdorff sense (cf. [7]).

Lemma 2.1. *Let* $X = \lim M_i$, $M_i \in \mathscr{M}(n, k, D)$. *Consider two normal minimal geodesics* $c_1 : [-\ell, \ell] \to X$, $c_2 : [0, \ell] \to X$ *with* $c_1(0) = c_2(0) = m$. *If* c_1, c_2 *are uniform limits of normal minimal geodesics* c_1^i, c_2^i *in* M_i, *then* $c : [-\ell, \ell] \to X$ *defined by* $c|_{[-\ell, 0]} = c_1|_{[-\ell, 0]}$, $c|_{[0, \ell]} = c_2$ *is minimal only if* $c_2 = c_1|_{[0, \ell]}$.

Proof. Suppose $c_1(t_0) \ne c_2(t_0)$ for some $t_0 \in (0, \ell)$. For i large, all $c_1^i : [-t_0, t_0] \to M_i$, $c_2^i : [0, t_0] \to M_i$ are defined and minimal. Let $\overline{c}_1^i$, $\overline{\overline{c}}_1^i$ be normal minimal geodesics in M_i from $c_2^i(0)$ to $c_1^i(-t_0)$, and to $C_1^i(t_0)$ respectively. Since $\lim c_2^i(0) = \lim c_1^i(0) = m$, we have $\lim L(\overline{c}_1^i) = \lim(\overline{\overline{c}}_1^i) = t_0$. By the Toponogov triangle comparison theorem, therefore, $\lim \theta_i = \pi$, where $\theta_i = \angle(\dot{\overline{c}}_1^i(0), \dot{\overline{\overline{c}}}_1^i(0))$. Since $\lim d(\overline{\overline{c}}_1^i(t_0), c_2^i(t_0)) = d(c_1(t_0), c_2(t_0)) > 0$, we conclude again, using the triangle comparison theorem, that $\angle(\dot{c}_2^i(0), \dot{\overline{\overline{c}}}_1^i(0)) \ge \alpha$, for some $\alpha > 0$ and all large i. Combining these facts we find $\beta < \pi$ such that $\angle(-\dot{\overline{c}}_1^i(0), \dot{c}_2^i(0))) \le \beta$ for i sufficiently large. The Toponogov distance comparison theorem then implies $d(\overline{c}_1^i(-t_0), c_2^i(t_0)) \le d$ for some $d < 2t_0$ and i large. In particular, $d(c_1(-t_0), c_2(t_0)) < 2t_0$.   q.e.d.

This lemma should be viewed as a uniqueness property for limits of minimal geodesics, they cannot "branch." This will play a crucial role in

understanding the geometry and topology of X. As a first example of this, we show that all minimal geodesics in X are limits of minimal geodesics in M_i.

Lemma 2.2. *Let* $X = \lim M_i$, $M_i \in \mathcal{M}(n, k, D)$ *and* $p = \lim p_i$ *and* $p \in X$, $p_i \in M_i$. *Then any minimal geodesic in* X *emanating from* p *is the limit (after possibly passing to a subsequence) of minimal geodesics in* M_i *emanating from* p_i.

Proof. Let $c\colon [0, 1] \to X$ be any minimal geodesic in X with $c(0) = p$ and say $c(1) = q$, $d(p, q) = L(c) = \ell$. For every m, there is an $i = i(m)$, and points $p_i^0, \cdots, p_i^m \in M_i$ so that $d(p_i^j, c(j/m)) \leq 2^{-m}$, $j = 0, \cdots, m$, and $p_i^0 = p_i$, $p_i^m = q_i$. Let $\gamma_i\colon [0, 1] \to M_i$ be a piecewise minimal geodesic in M_i from p_i via the p_i^j's to q_i. Clearly $\lim L(\gamma_i) = \ell$ and γ_i converges to c uniformly.

Fix $0 < s < t < 1$ arbitrarily. First choose minimal geodesics $c_s^i\colon [0, s] \to M_i$, $\bar{c}_s^i\colon [s, 1] \to M_i$ from $\gamma_i(0)$ to $\gamma_i(s)$, and from $\gamma_i(s)$ to $\gamma_i(1)$. We may assume (after again possibly passing to a subsequence) that c_s^i, $\bar{c}_s^i$ converges to minimal geodesics $c_s\colon [0, s] \to X$, $\bar{c}_s\colon [s, 1] \to X$ with $c_s(0) = c(0)$, $c_s(s) = \bar{c}_s(s) = c(s)$, and $\bar{c}_s(1) = c(1)$. In particular, $L(c_s^i) + L(\bar{c}_s^i) \to L(c)$. From the Toponogov comparison theorem we first conclude that $\measuredangle (-\dot{c}_s^i(s), \cdots, \dot{\bar{c}}_s^i(s)) \to \pi$ and then that the unique (possibly nonminimal) geodesic extension $c_s^i\colon [s, 1] \to M_i$ of $c_s^i\colon [0, s] \to M_i$ converges uniformly to $\bar{c}_s = \lim \bar{c}_s^i$. Thus, there is a minimal geodesic $c_s\colon [0, 1] \to X$ from $c(0)$ through $c(s)$ to $c(1)$ which is the limit of geodesics $c_s^i\colon [0, 1] \to M_i$ that are minimal when restricted to $[0, s]$. Observe that by similarly extending $\bar{c}_s^i\colon [s, 1] \to M_i$ backwards to a possibly nonminimal geodesic $\bar{c}_s^i\colon [0, 1] \to M_i$, we obtain $\bar{c}_s = c_s$ where $\bar{c}_s = \lim \bar{c}_s^i$. Now repeat the argument with s replaced by t. We will now show that the minimal geodesic $c_t\colon [0, 1] \to X$ obtained this way satisfies $c_t|_{[0, s]} = c_s|_{[0, s]}$. Since $c_t|_{[0, t]}$ and $c_s|_{[0, s]}$ are both limits of minimal geodesics, we only need to show $c_t(s) = c_s(s)$ according to Lemma 2.1. For this, let $c_{st}^i\colon [s, t] \to M_i$ be a minimal geodesic from $\gamma_i(s)$ to $\gamma_i(t)$. The arguments above applied to c_t^i and $\bar{c}_t^i$ yield $\lim \bar{c}_t^i(s) = \lim c_{st}^i(s) = c(s)$. On the other hand, we already know that $c_t(s) = \lim c_t^i(s) = \lim \bar{c}_t^i(s)$.

For each m, using the above construction $(m - 1)$ times, we obtain a minimal geodesic $c^m\colon [0, 1] \to X$ for which $c^m(j/m) = c(j/m)$, $j = 0, \cdots, m$, and c^m is the limit of geodesics emanating from p_i in M_i, each of which is minimal when restricted to $[0, (m - 1)/m]$. A standard

diagonal argument now implies that c is a limit of minimal geodesics in M_i emanating at p_i.   q.e.d.

Note that if c is a minimal geodesic in X from $p = \lim p_i$ to $q = \lim q_i$, p_i, $q_i \in M_i$, then in general c is not the limit of minimal geodesics c_i in M_i from p_i to q_i. In particular, geodesic triangles in X are generally not limits of geodesic triangles in M_i.

Now let (p, c_1, c_2) be a hinge in X at $p \in X$, i.e., $c_i: [0, 1] \to X$ are minimal geodesics in X emanating from p with lengths $L(c_i) = \ell_i$, $i = 1, 2$. By Lemma 2.2, (p, c_1, c_2) is the limit of geodesic hinges (p_i, c_1^i, c_2^i) in M_i. In particular, $d(c_1(1), c_2(1)) = \lim d(c_1^i(1), c_2^i(1))$. Using the Toponogov triangle comparison theorem, we find for each (p_i, c_1^i, c_2^i) a hinge $(\overline{p}_i, \overline{c}_1^i, \overline{c}_2^i)$ in $\overline{M}^2$ with $d(c_1^i(1), c_2^i(1)) = d(\overline{c}_1^i(1), \overline{c}_2^i(1))$. Moreover, for any fixed s, $t \in [0, 1]$, we have $d(c_1^i(s), c_2^i(t)) \geq d(\overline{c}_1^i(s), \overline{c}_2^i(t))$ for all i. Clearly we may assume that $(\overline{p}_i, \overline{c}_1^i, \overline{c}_2^i)$ converges to a hinge $(\overline{p}, \overline{c}_1, \overline{c}_2)$ in $\overline{M}^2$ with $L(\overline{c}_i) = L(c_i)$, $i = 1, 2$, and $d(\overline{c}_1(1), \overline{c}_2(1)) = d(c_1(1), c_2(1))$. Furthermore, $d(c_1(s), c_2(t)) \geq d(\overline{c}_1(s), \overline{c}_2(t))$ for any s, $t \in [0, 1]$. In this sense, the Toponogov comparison theorem is valid in X. Therefore we say that X has Toponogov curvature $\geq k$ (cf. Theorem A(i)). For the local version of this, cf., e.g., [20].

Proposition 2.3. *Let* $X = \lim M_i$, $M_i \in \mathcal{M}(n, k, D)$, *and* $D \leq \pi/K_M$, *when* $k > 0$. *Then the following hold:*

(i) *For any* $p \in X$ *there exist closed subsets* $C(\overline{p}) \subset \overline{B}_k^n(\overline{p}, D)$ *together with surjective distance nonincreasing maps* $\exp_p: C(\overline{p}) \to X$. *Moreover,* $C(\overline{p})(C(\overline{p}) \cap B_k^n(\pi/\sqrt{k})$ *when* $k > 0)$ *is star-shaped around* $\overline{p}$ *and* $\exp_p$ *maps geodesic segments emanating from* $\overline{p}$ *to minimal geodesics in* X *emanating at* p. *Conversely any minimal geodesic emanating from* p *is the image under* $\exp_p$ *of a geodesic segment emanating from* $\overline{p}$.

(ii) X *has dimension* $\leq n$. *In fact, the Hausdorff dimension of* X *is* $\leq n$ *since* $m_n(X) \leq m_n(\overline{B}_k^n(D)) < \infty$.

(iii) *If* $\dim X = n$, *every* $C(\overline{p})$ *has nonempty interior.*

Proof. (i) Fix $p \in X$ and a sequence $\{p_i\}$, $p_i \in M_i$ with $\lim p_i = p$. Consider the sequence $\exp_{p_i}: C(p_i) \to M_i$ of exponential maps, where $C(p_i) \subset T_{p_i} M_i$ is the tangent cut locus of p_i together with its interior. Endow the open D-ball in $T_{p_i} M_i$ with a Riemannian metric of constant curvature k. Viewing $C(p_i)$ as a metric subspace of $\overline{B}_k^n(\overline{p}, D)$ the map $\exp_{p_i}$ is distance nonincreasing according to the Toponogov distance comparison theorem. Since the space of closed subsets of a compact metric space is compact with respect to the classical Hausdorff metric, we may assume,
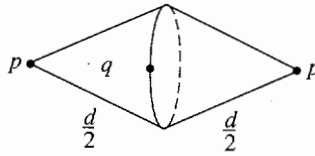
FIGURE 2

by possibly passing to a subsequence, that $\{C(p_i)\}$ converges to a closed subset $C(\overline{p}) \subset \overline{B}_k(\overline{p}, D)$. Furthermore, an Arzela-Ascoli type argument (cf. appendix) shows that after again possibly passing to a subsequence, the maps $\exp_{p_i} \colon C(p_i) \to M_i$ converge to a map $\exp_p \colon C(\overline{p}) \to X$, i.e., for every $v \in C(\overline{p})$ and sequence $\{v_i\}$, $v_i \in C(p_i)$ with $\lim v_i = v$, we have $\lim \exp_{p_i}(v_i) = \exp_p(v)$. By construction $C(\overline{p})$ is star-shaped around $\overline{p}$ (expect in case $k > 0$ and $-p \in C(\overline{p})$, where only $C(p) \cap B(\overline{p}, \pi/\sqrt{k})$ is star-shaped), and $\exp_p \colon C(\overline{p}) \to X$ is surjective and distance nonincreasing. Also, since each $\exp_{p_i}$ is a radial isometry, so is $\exp_p$ (restricted to $C(\overline{p}) \cap B(\overline{p}, \pi/\sqrt{k})$ when $k > 0$). The converse is now a consequence of 2.2.

(ii) This follows immediately from [16] since $\exp_p \colon C(\overline{p}) \to X$ is surjective and distance nonincreasing.

(iii) If $\dim X = n$, then also $\dim C(\overline{p}) = n$, since $\exp_p$ is surjective and distance nonincreasing. In particular $C(\overline{p})$ has nonempty interior [16].   q.e.d.

The exponential maps constructed above are by no means unique.

Examples 2.2.   (a) Clearly the suspension $X = C_+ \cup C_-$, where $C_\pm$ are flat Euclidean cones (Figure 2), is the limit of Riemannian manifolds $M_i = (S^2, g_i)$ with $K_{M_i} \geq 0$, $d_{M_i} \leq D$, and $V_{M_i} \geq v$. At the points $p \in X$ we have the cut locus pictures (Figure 3, next page), where $\exp_p$ of the local cut locus picture of the left identifies opposite sides. At the point $q \in X$ one gets Figure 4 (next page).

(b) Shortening the meridian circles by the factor i^{-1} provides a sequence $M_i = (S^2, g_i)$ of Riemannian manifolds with $K_{M_i} > 0$, $d_{M_i} \leq D$, and $\lim M_i = X$ is an interval. For an endpoint p of the interval, the typical $C(\overline{p})$ is also an interval, but as in (a) it is also possible for it to be a 2-disc.

Remark 2.3.   The statement (ii) in 2.3 remains true if the lower sectional curvature bound is replaced by a lower bound for the Ricci curvature (cf. [18]).
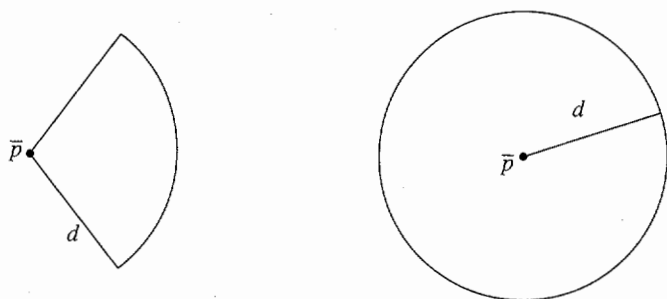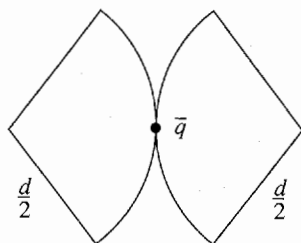
FIGURE 3

FIGURE 4

We are now ready to complete the:

Proof of Theorem A. (ii). Suppose $\dim X = m$, but X has dimension $\leq m - 1$ at some point $p \in X$. By definition there is a neighborhood U of p such that $\dim \partial \leq m - 2$ and $\dim X - \overline{U} = m$. Consider the set Γ_{pX} of minimal geodesics emanating from p. Define the relation R in $(X - \overline{U}) \times \partial U$ by xRy if and only if $y \in \partial U$ lies on a geodesic in Γ_{px}. This relation is closed and continuous; i.e., for every closed set $F \subset X - \overline{U}$, the set $\{y \in \partial U | x \in F, xRy\}$ is closed, and for every open set $G \subset \partial U$, the set $\{x \in X - \overline{U} | y \in G, xRy\}$ is open. From the proof of Theorem VI.7 in §4 of [16], we conclude the existence of a point $y \in \partial U$ so that $\dim\{x \in X - \overline{U} | xRy\} \geq 2$. Since geodesics are 1-dimensional, there are clearly distinct minimal geodesics $c_1: [0, 1] \to X$, $c_2: [0, 1] \to X$ with $c_1(0) = c_2(0) = p$, $c_1(\ell_1) = y = c_2(\ell_2)$, and $c_1(1) \neq c_2(1)$. This contradicts 2.1 (2.2), and hence X has dimension m at all points.

(iii) Next, let O be a nonempty open subset of X with $\overline{O} \neq X$. From the above, the open set $X - \overline{O}$ is m-dimensional. If $p \in O$, we can

let O play the role of U from above, and the argument there then gives $\dim \partial O \geq m - 1$.

(iv) $X = X_1 \cup X_2$, $X_i \neq X$ closed nonempty. To show that X is a Cantor manifold, we must verify that $\dim X_1 \cap X_2 \geq m - 1$. Now $O_1 = X - X_2$ is open, nonempty, and $\overline{O}_1 \subset X_1 \neq X$. By (ii) $\dim O_1 \geq m - 1$ and since $\partial O_1 \subset X_1 \cap X_2$, this completes the proof.

Remark 2.4. The proof above shows that any locally compact inner metric space X with Toponogov curvature $\geq k$ satisfies (ii) to (iv) of Theorem A. We expect the structure of such spaces to be much more restrictive, however. In particular, the only 1-dimensional examples are circles and intervals. Theorem A also holds for limit spaces of complete open manifolds in the pointed Gromov-Hausdorff Topology.

Although the Toponogov triangle comparison theorem is true for limit spaces in our class, the following example indicates restrictions on its use to non-Riemannian spaces (see however [13]).

Example 2.5. (cf. also [23]) Let M be a closed Riemannian manifold, and G a compact group of isometries on M. The orbit space $X = M/G$ has an induced length metric (also called inner metric) with curvature bounded from below by $\min K_M$ in the sense of Toponogov. Note that X is the limit of orbifolds M/G_i, where $G_i \to G$ are finite subgroups of G. In particular, the suspension $\sum CP^n = S^{2n+2}/S^1$ carries a length space structure with curvature ≥ 1 and diameter π. Observe that $d_X \leq \pi$ for any length space X, which satisfies Toponogov's comparison theorem with $K_X \geq 1$. The above example therefore shows that the Toponogov maximal diameter theorem fails in a non-Riemannian setting. Furthermore, when $n = 0$, $[0, \pi] = \sum CP^0 = S^2/S^1 = \lim S^2/\mathbf{Z}_i$. By smoothing out the two singularities on $S^2/\mathbf{Z}_i$, we get a sequence $M_i = (S^2, g_i)$ of Riemannian manifolds with $K_{M_i} \geq 1$, $V_{M_i} \to 0$, although $d_{M_i} \to \pi$.

3. Nonexistence of voluminous manifolds

In this section we will investigate the possible existence of manifolds M^n with $K_M \geq k$, $d_M \leq D$, and almost maximal volume $V(n, k, D)$.

To this end, suppose $\{M_i\}$ is a sequence of closed, connected Riemannian n-manifolds with $K_{M_i} \geq k$, $d_{M_i} \leq D$, and $V_{M_i} \to V(n, k, D)$. By the Gromov precompactness theorem [7], we may assume that M_i converges in the Gromov-Hausdorff metric to a space X. We proceed to show that X is a Riemannian n-manifold with constant curvature k, $d_X = D$,

and $V_X = V(n, k, D)$. The only manifolds of this kind are the sphere and the real projective space of constant curvature $k > 0$. The proof of the first part of Theorem B will then be complete.

Lemma 3.1. *For $p \in X$ any $C(\overline{p}) = \overline{B}_k^n(\overline{p}, D)$, $\exp_p : B_k^n(\overline{p}, D) \to X$ is injective, and $\exp_p : \overline{B}_k^n(\overline{p}, \frac{D}{2}) \to \overline{B}(p, \frac{1}{2}D)$ is a bijective isometry.*

Proof. As in the proof of 2.3, choose $p_i \in M_i$ so that $p_i \to p$ and $C(p_i) \to C(\overline{p})$ inside $\overline{B}_k^n(\overline{p}, D)$. Now since $V_{M_i} \leq \mathrm{vol}\, C(p_i) \leq V(n, k, D)$ and $V_{M_i} \to V(n, k, D)$, we conclude that $\overline{B}_k^n(\overline{p}, D) - C(\overline{p})$ does not have interior points, and therefore it is empty.

The two statements about the exponential map are analogous applications of 1.1 (cf. also [11]). We therefore confine our attention to the latter.

As $\exp_p : C(\overline{p}) \cap \overline{B}_k^n(\overline{p}, R) \to \overline{B}(p, R)$ is already surjective and distance nonincreasing for any $R \leq D$, it suffices to show that it does not decrease any distances. Thus assume there are u, $v \in \overline{B}_k^n(\overline{p}, \frac{1}{2}D)$ with

$$d(\exp_p u, \exp_p v) < d(u, v) - \delta$$

for some $\delta > 0$. Select u_i, $v_i \in C(p_i) \cap \overline{B}_k^n(\overline{p}, \frac{1}{2}D)$ so that $u = \lim u_i$, $v = \lim v_i$. For i sufficiently large, $d(\exp_p u_i, exp_p v_i) \leq d(u_i, v_i) - \delta$. Now define $Q_i = \{\exp_{p_i} u_i, \exp_{p_i} v_i\} \subset M_i$, $F_i(q, t) = \frac{1}{2}d(u_i, v_i) \leq \frac{1}{2}D$, and $G_i(q, t) = t$ for $(q, t) \in Q_i \times [0, \infty)$. By Lemma 1.1 (cf. 1.2(b), (d)) we have

$$\mathrm{vol}(M_i - B(\exp_{p_i} u_i, \tfrac{1}{2}d(u_i, v_i)) \cup B(\exp_{p_i} v_i, \tfrac{1}{2}d(u_i, v_i)))$$
$$\leq V(n, k, D) - 2V(n, k, \tfrac{1}{2}d(u_i, v_i)),$$

and

$$\mathrm{vol}(B(\exp_{p_i} u_i, \tfrac{1}{2}d(u_i, v_i)) \cup B(\exp_{p_i} v_i, \tfrac{1}{2}d(u_i, v_i)))$$
$$\leq \mathrm{vol}(B_k^n(\tilde{u}_i, \tfrac{1}{2}d(u_i, v_i)) \cup B_k^n(\tilde{v}_i, \tfrac{1}{2}d(u_i, v_i))),$$

where $d(\tilde{u}_i, \tilde{v}_i) = d(u_i, v_i) - \delta$. Hence

$$V_{M_i} \leq V(n, k, D) - \mathrm{vol}(B_k^n(\tilde{u}_i, \frac{1}{2}d(u_i, v_i)) \cap B_k^n(\tilde{v}_i, \frac{1}{2}d(u_i, v_i))),$$

and therefore,

$$\lim V_{M_i} \leq V(n, k, D) - V(n, k, \frac{\delta}{2}).$$

This contradicts $V_{M_i} \to V(n, k, D)$. q.e.d.

It follows directly from 3.1 that X is a Riemannian n-manifold of constant curvature k, and injectivity radius=diameter=D. Thus, $\mathrm{vol}\, X = V(n, k, D)$. q.e.d.

To complete the proof of Theorem B, suppose $X = \lim M_i$ where $K_{M_i} \geq 1$, $d_{M_i} = D \in [\pi/2, \pi]$, $V_{M_i} \to \frac{D}{\pi} V(n, 1, \pi)$, and $M_i \neq \mathbf{R}P^n$. Let p_0, $p_1 \in X$ be points with $d(p_0, p_1) = d_X = D$ and consider $H_\ell = \{x \in X | d(x, p_\ell) \leq d(x, p_{\ell+1})\}$, indices mod 2, together with the set $C = \{x \in X | d(x, p_\ell) = \pi/2, \ell = 0, 1\}$ which is convex by the Toponogov comparison theorem. Clearly $X = H_0 \cup H_1$ and C is a subset of $E = H_0 \cap H_1 = \{x \in X | d(x, p_0) = d(x, p_1)\} = \partial H_\ell$, $\ell = 0, 1$. We proceed to show that $X - C$ is a smooth manifold of constant curvature 1. In fact each $H_\ell - C$, $\ell = 0, 1$ is intrinsically isometric to the convex set $H(\overline{p}, D) - S_p^{n-2}$, where $H(\overline{p}, D) = \{\overline{x} \in S^n(1) | d(\overline{x}, \overline{p}) \leq d(\overline{x}, \{\overline{q}, \overline{r}\})\}$, $S_p^{n-2} = H(\overline{p}, D) \cap \{\overline{x} \in S^n(1) | d(\overline{x}, \overline{p}) = \pi/2\}$, and $\overline{p}$, $\overline{q}$, $\overline{r}$ all lie on the same great circle with $d(\overline{p}, \overline{q}) = d(\overline{p}, \overline{r}) = D$ (cf. also the proof of Theorem 1.4).

Lemma 3.2. *For any* $C(\overline{p}_\ell) \subset \overline{B}_1^n(\overline{p}_\ell, D)$ *there are unique* $H(\overline{p}_\ell, D)$ $\subset C(\overline{p}_\ell)$ *such that* $\exp_{p_\ell} : H(\overline{p}_\ell, D) - S_\ell^{n-2} \to X$ *are injective locally isometric embeddings,* $\ell = 0, 1$. *This in turn is only possibly in the trivial case,* $D = \pi$.

Proof. Choose $p_\ell^i \in M_i$ so that $\lim p_\ell^i = p_\ell$, $\ell = 0, 1$. For each i, $M_i = H(p_0^i) \cup H(p_1^i)$, $V_{M_i} = V_{H(p_0^i)} + V_{H(p_1^i)}$, and $V_{H(p_\ell^i)} \leq V_{H(\overline{p}, D)}$ by 1.1 (cf. 1.2(c)). Since $V_{M_i} \to 2V_{H(\overline{p}, D)}$, we conclude $V_{H(p_\ell^i)} \to V_{H(\overline{p}, D)}$, $\ell = 0, 1$. In particular, using the appendix of [10], we may assume that $\dot{\Gamma}_{p_\ell^i p_{\ell+1}^i} \subset T_{p_\ell^i}^{M_i}$ converges to some $\dot{\Gamma}_{\overline{p}_\ell \{\overline{q}_\ell, \overline{r}_\ell\}}$, with $\overline{p}_\ell$, $\overline{q}_\ell$, $\overline{r}_\ell \in C(\overline{p}_\ell) \subset B_1^n(\overline{p}_\ell, D) \subset S^n(1)$ as described above, $\ell = 0, 1$. Let $H(\overline{p}_\ell, D)$ be the corresponding convex sets defined above. Now since $V_{H(p_\ell^i)} \to V_{H(\overline{p}_\ell, D)}$ we conclude, as in the proof of 3.1, that $H(\overline{p}_\ell, D) - C(\overline{p}_\ell, D)$ have no interior points, i.e., $H(\overline{p}_\ell, D) \subset C(\overline{p}_\ell, D)$, $l = 0, 1$. Moreover $\exp_{p_l} : \overline{H}(\overline{p}_l, D) \to H(p_l)$ is surjective since $\exp_{p_\ell} : C(\overline{p}_\ell) \to X$ is surjective, $l = 0, 1$.

Now let $D_{\overline{q}_\ell}^{n-1} = \{\overline{x} \in H(\overline{p}_\ell, D) | d(x, \overline{p}_\ell) = d(x, \overline{q}_\ell)\}$ and similarly define $D_{\overline{r}_\ell}^{n-1}$, $\ell = 0, 1$. Then $S_{\overline{p}_\ell}^{n-2} = D_{\overline{q}_\ell}^{n-1} \cap S_{\overline{r}_\ell}^{n-1}$ and $\exp_p(S_{\overline{p}_\ell}^{n-2}) = \exp_{p_{\ell+1}}(S_{\overline{p}_{\ell+1}}^{n-2}) = C$. Assume that $\overline{q}_\ell$, $\overline{r}_\ell = 0, 1$, are chosen so that $\exp_{p_0}(D_{\overline{q}_0}^{n-1}) = \exp_{p_1}(D_{\overline{q}_1}^{n-1}) =: E_q \subset E$, and $\exp_{p_0}(D_{\overline{r}_0}^{n-1}) = \exp_{p_1}(D_{\overline{r}_1}^{n-1}) =: E_r \subset E$. With this notation, $E = E_1 \cup E_r = H_0 \cap H_1$ and $C = E_q \cap E_r$. To

show that $\exp_{p_\ell}$ restricted to $\mathrm{int}(H(\overline{p}_\ell, D)$ is an injective local isometry is in essence identical to the argument given in 3.1. The only difference is that we use 1.1 in versions (1.2) (b) and (c) combined, i.e., we estimate the volume of complements of balls in half spaces.

In the remaining part of the proof we need to use both exponential maps simultaneously. Suppose, e.g., $\exp_{p_0}(u) = \exp_{p_0}(v) = x \in E - C$ and choose w so that $\exp_{p_1}(w) = x$. Pick a positive $\varepsilon < \min\{\frac{1}{2}d(u, v)$, $d(u, S_{p_0}^{n-2})$, $d(v, S_{p_0}^{n-2})$, $d(w, S_{p_1}^{n-2})\}$. A limit argument based on 1.1 will now yield $\lim V_{M_i} \leq 2V_{H(\overline{p}, D)} - \frac{1}{2}V_{B_1^n(\varepsilon)}$, i.e., $\exp_{p_\ell}$ restricted to $\partial H(\overline{p}_\ell, D) - S_{\overline{p}_\ell}^{n-2}$ is injective. The same modification of the argument in 3.1 now also implies that these restrictions are local isometries.

With the notation introduced above, therefore, $A = \exp_{p_1}^{-1} \circ \exp_{p_0}$: $\mathrm{int}\, D_{q_0}^{n-1} \to \mathrm{int}\, D_{q_1}^{n-1}$, $B = \exp_{p_0}^{-1} \circ \exp_{p_1}$: $\mathrm{int}\, D_{r_1}^{n-1} \to \mathrm{int}\, D_{r_0}^{n-1}$ are bijective local, and hence global isometries. Now consider the isometry $F = B \circ R_{\overline{p}_1} \circ A \circ R_{\overline{p}_0}$: $\mathrm{int}\, D_{r_0}^{n-1} \to \mathrm{int}\, D_{r_0}^{n-1}$, where $R_{\overline{p}_\ell}$, $\ell = 0, 1$ are the obvious rotations in $S^n(1)$ which fixes $S_{\overline{p}_\ell}^{n-2}$. Then for any $u \in S_{\overline{p}_0}^{n-2} = \partial D_{r_0}^{n-1} = \partial D_{q_0}^{n-1}$, clearly $y = \exp_{p_0}(u) = \exp_{p_0}(F(u))$ by continuity, and $y = \exp_{p_1}(R_{\overline{p}_0}(A(u)) = \exp_{p_1}(R_{\overline{p}_0}(A(f(u)))$. Note that $\measuredangle (D_{r_\ell}^{n-1}, D_{q_\ell}^{n-1}) \geq \pi/2$, since $D \geq \pi/2$, i.e., balls in $H(\overline{p}_\ell, D)$ centered at points in $S_{\overline{p}_\ell}^{n-2} = \partial D_{q_\ell}^{n-1} \cap \partial D_{r_\ell}^{n-1}$ have volume $\geq \frac{1}{4}$ th the volume of balls in $S^n(1)$. Another limit argument then shows that $F = id$ or if $D = \pi/2$ possibly $F = -id$. In the latter case we get an induced surjective map $\mathbf{R}P^n = H(p_0, \pi/2) \cup H(p_1, \pi/2)/ \sim \to X$. This is impossible since each $M_i \neq \mathbf{R}P^n$ is homeomorphic to S^n [5] and hence X has the homotopy type of S^n [19]. In all cases therefore, $F = id$. Thus for any $u \in D_{q_0}^{n-1}$ and $\tilde{u} \in D_{r_0}^{n-1}$, the opposite point relative to $\overline{p}_0$, we have $\exp_{p_0}(u) = \exp_{p_1}(A(u))$ and $\exp_{p_0}(\tilde{u}) = \exp_{p_1}(\widetilde{A(u)})$. Now choose u so that, e.g., $\frac{\delta}{2} = d(u, p_0) = d(u, S_{\overline{p}_0}^{n-2}) > \frac{D}{2}$, if $D < \pi$. Then the $\frac{\delta}{2}$-balls in $H(\overline{p}_\ell, D)$ centered at u, $\tilde{u}$, $A(u)$, $\widetilde{A(u)}$ are disjoint. A limit volume argument as before, then implies $d(\exp_{p_0}(u), \exp_{p_0}(\tilde{u})) \geq \delta > D$ which contradicts $d_X \leq D$.

Remark 3.3.   We observe now that when X is the real projective space of constant curvature $k > 0$, then M_i is homeomorphic to X for i sufficiently large, at least in dimensions ≥ 5. In [19] maps $f_i: X \to M_i$ with $\mathrm{diam}(f_i) = \sup\{\mathrm{diam}\, f_i^{-1}(x) | x \in X\} \to 0$ as $i \to \infty$ are exhibited.

Since M_i and X are manifolds of dimension ≥ 5, a result of Ferry [4] implies that the maps f_i are homotopic to homeomorphisms for large i.

Appendix: A Generalization of the Arzela-Ascoli Theorem

Let $\{X_i\}$, $\{Y_i\}$ be sequences of compact metric spaces. A sequence $\{f_i\}$ of maps $f_i\colon X_i \to Y_i$ is said to be equicontinuous if for any $\varepsilon > 0$, there is a $\delta > 0$ such that $f_i(B(x_i, \delta)) \subset B(f_i(x_i), \varepsilon)$ for all $x_i \in X_i$, and $i = 1, 2, 3, \cdots$.

Suppose X, Y are compact metric spaces and $X = \lim X_i$, $Y = \lim Y_i$ with respect to the Gromov-Hausdorff metric. Extend the metrics on X, X_i to a metric on $X \coprod X_i$ so that $X_i \to X$ inside $X \coprod X_i$ in the classical Hausdorff sense, and similarly for Y, Y_i (cf. [7]). We say that $f_i\colon X_i \to Y_i$ converges to $f\colon X \to Y$ provided $f_i(x_i) \to f(x)$ whenever $x_i \to x$. With this terminology we have:

Theorem.   *For any equicontinuous sequence of maps* $\{f_i\}\colon \{X_i\} \to \{Y_i\}$ *one can extract a convergent subsequence, converging to a continuous map* $f\colon X \to Y$.

With the necessary changes, this is demonstrated as the classical theorem (cf. also [7] in the case where all maps are isometries).

References

[1]   A. D. Alexandrov, *Die innere Geometrie des Konvexen flächen*, Akademic-Verlag, Berlin, 1955.

[2]   J. Cheeger & D. G. Egin, *Comparison theorems in Riemannian geometry*, North-Holland Math. Library, **9**, North-Holland, Amsterdam, 1975.

[3]   O. Durumeric, *Manifolds of almost half of the maximal volume*, Proc. Amer. Math. Soc. **104** (1988) 277–283.

[4]   S. Ferry, *Homotoping ε-maps to homeomorphisms*, Amer. J. Math. **101** (1979) 567–582.

[5]   D. Gromoll & K. Grove, *A generalization of Berger's rigidity theorem for positively curved manifolds*, Ann. Scient. École Norm. Sup. **20** (1987) 227–239.

[6]   M. Gromov, *Curvature, diameter and Betti numbers*, Comment. Math. Helv. **56** (1981) 179–195.

[7]   ――――, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. **53** (1981) 53–73.

[8]   M. Gromov, J. Lafontaine & P. Pansu, *Structure métrique pour les variétés Riemanniennes*, Cedic/Fernant Nathan, Paris, 1981.

[9]   K. Grove, *Metric differential geometry*, Differential Geometry, Proc. Nordic Summer School, Lyngby 1985 (ed. V. L. Hansen), Lecture Notes in Math., Vol. 1263, Springer, 1987, 171–227.

[10] K. Grove & P. Petersen V, *Bounding homotopy types by geometry*, Ann. of Math. **128** (1988) 195–206.

[11] ――――, *Homotopy types of positively curved manifolds with large volume*, Amer. J. Math. **110** (1988) 1183–1188.

[12] ____, *A pinching theorem for homotopy spheres*, J. Amer. Math. Soc. **3** (1990).

[13] ____, *Excess and rigidity of inner metric spaces*, Preprint.

[14] K. Grove & K. Shiohama, *A. generalized sphere theorem*, Ann. of Math. **106** (1977) 201–211.

[15] K. Grove, P. Petersen V, & J. Y. Wu, *Geometric finiteness theorems via controlled topology*, Invent Math. **99** (1990) 205–213.

[16] W. Hurewicz & H. Wallman, *Dimension theory*, Princeton University Press, Princeton, NJ, 1941.

[17] Y. Otsu, K. Shiohama & T. Yamaguchi, *A new version of differentiable sphere theorem*, Invent. Math. **98** (1989) 219–228.

[18] P. Petersen V, *The fundamental group of almost non-negatively curved manifolds*, Preprint.

[19] ____, *A finiteness theorem for metric spaces*, J. Differential Geometry **31** (1990) 387–395.

[20] W. Rinow, *Die Innere Geometrie der Metrischen Räume*, Springer, Berlin, 1961.

[21] J. Y. Wu, *A volume/diameter-ratio for positively curved manifolds*, Michigan Math. J., to appear.

[22] T. Yamaguchi, *Lipschitz convergence of manifolds of positive Ricci curvature with large volume*, Math. Ann. **284** (1989) 423–436.

[23] ____, *Collapsing and pinching under a lower curvature bound*, Ann. of Math., to appear.

UNIVERSITY OF MARYLAND
PRINCETON UNIVERSITY